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Distribution of the number of fragmentations in continuous fragmentation

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Abstract. The continuous fragmentation equation is generalized to include the number of fragmentations as a variable. The fragmentation rate $\alpha(n, x)$ and the probability for the production of progeny size y from parent size x, $B_n(x \to y)$, are both made dependent on the number n of fragmentations experienced by the fragment during its history. An interpretation where n is the number of impacts experienced by the fragment is possible by also allowing a single progeny equalling the parent in size. Solutions for the cases where $\alpha(n)$ does not depend on x are studied. The theory is adapted to model clustering in fragmentation-generated cellular structures.

1. Introduction

The linear fragmentation equation for the size distribution $f(x, t)$ of fragments with maximum size 1 is usually written

$$\frac{\partial}{\partial t} N f(x) = N \int_x^1 \mathrm{d}y\, \alpha(y) f(y) B(y \to x)\kappa(y) - N\alpha(x) f(x) \tag{1}$$

where $N(t)$ is the total number of fragments, $\alpha(x, t)$ is the fragmentation rate, and $B(y \to x; t)\,\mathrm{d}x$ is the probability for a fragmentation event with parent size y to produce progeny size x. The expectation of $B(y \to x; t)$ is $y/\kappa(y, t)$ where $\kappa(y, t)$ is the average number of progeny for parent size y. For q-ary fragmentation $\kappa = q$. Equation (1) and its asymptotic form, the scaling equation, have been extensively studied for $\alpha(x) \sim x^\rho$, where ρ is any real number, and solutions for various $B(y \to x)$ constructed [2–8, 11–14, 16]. These solutions can be used to describe fragmentation processes where $N(t)$ is large so that the continuous formalism can be assumed to apply, that is, either in cases where $N(0)$ is large or asymptotically for large t in cases where $N(0)$ is small. If the number of fragments is small discrete and combinatorial approaches are required [1, 4, 9, 10].

Here, the fragmentation number n is defined as the number of fragmentation events experienced by a fragment and it is introduced as a variable to the continuous fragmentation formalism. Rather than an additional parameter, the fragmentation number is a conjugate description to that referring to size. If at $t = 0$ the value of n for all fragments in the initial configuration is known, the state (n, x) is unambiguously definable for the fragments for all subsequent times. If $N(t)$ is large then by monitoring for a short time $[t, t + \Delta t]$ all n-fragments, all x-fragments, or all (n, x)-fragments the fragmentation rates $\alpha(n, t), \alpha(x, t)$ and $\alpha(n, x, t)$ can be determined. The process can be described in terms of any of these.

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However, the (n, x)-description includes the other two. In a fragmentation event the x-fragments produce y-fragments, $y < x$, while the n-fragments always produce $(n + 1)$-fragments so that (n, x)-fragments produce $(n + 1, y)$-fragments. Thus, the complete continuous description of the process can be based on the fragmentation rate $\alpha(n, x, t)$ and the progeny distribution $B_n(x \to y; t)$ which are all allowed to depend on n, x and t.

A possible physical motivation to consider the (n, x)-state is to describe processes during which the fragments grow weaker or stronger. The first could follow from the loosening of the composition of the fragments owing to repeated fragmentations, and the latter from the toughening of the fragments by impacts, or because most faults initiating the fragmentation are already used up. This is more obvious by allowing

$$B_n(x \to y; t) = (1 - \nu(n, x, t))\delta(x - y) + \nu(n, x, t)B_n^*(x \to y : t). \tag{2}$$

Here n can be interpreted as the impact number (number of impacts experienced by a fragment), in which case $\alpha(n, x, t)$ is the impact rate and $\nu(n, y, t)$ the probability for the fragment of size x and having suffered n impacts to fragment in the next impact. This is close to what actually happens in many forced fragmentation processes (grinding mills) and in collision-driven fragmentation. By a suitable choice of $\alpha(n, x, t)$ and $\nu(n, y, t)$ it is possible, for example, to describe processes where fragments that have suffered many impacts without breaking will probably remain unbroken in further impacts, or where further fragmenting ceases for small fragments. As the formalism does not change, the term fragmentation number is used in the following but the possibility of the impact number interpretation is kept in mind.

The introduction of the fragmentation number can be exemplified by studying (1) for q-ary fragmentation $\kappa = q$ in the special case $\alpha = 1$ and $B(y \to x)\,\mathrm{d}x \equiv B(x/y)(\mathrm{d}x/y)$, where $B(z)\,\mathrm{d}z$ is a probability distribution defined for $0 \leqslant y \leqslant 1$ and with the expectation $1/q$. The transform

$$F(s, t) = \int_0^1 \mathrm{d}x\, x^S f(x, t) \qquad D(s) = \int_0^1 \mathrm{d}x\, x^S B(x) \tag{3}$$

reduces (1) to

$$\frac{\partial}{\partial t} NF(s, t) = NF(s, t)(qD(s) - 1) \tag{4}$$

where $F(0, t) = D(0) = 1$ and $q = 1/D(1)$. For an initial configuration at $t = 0$ where a large number N_0 of fragments follows distribution $f_0(x)$ with the transform $F_0(s)$ the solution of (4) is

$$F(s, t) = F_0(s)\mathrm{e}^{q(D(s)-1)t} \qquad N(t) = N_0\mathrm{e}^{(q-1)t}. \tag{5}$$

The exponential in $F(s, t)$ can be formally written as a series from which the solution of (1) is obtained as

$$f(x, t) = \mathrm{e}^{-qt} f_0(x) + f_0(x) * \mathrm{e}^{-qt} \sum_{n=1}^{\infty} \overbrace{[B * B * \cdots * B]}^{n}(x) \frac{(qt)^n}{n!} \tag{6}$$

where $[f_1 * f_2](x) = \int_x^1 \mathrm{d}y\, f_1(y) f_2(x/y)(1/y)$. Expression (6) is the probability that a fragment randomly chosen from the N fragments at time t has size x. The first term describes the probability that this is due to some fragment of the initial configuration that has persisted. In the other terms $f_0(x) * \overbrace{[B * \cdots B * B]}^{n}(x)$ is the size distribution for particles that have fragmented n times and the remaining factor in these terms is the probability that a fragment randomly chosen at time t has fragmented n times. For $B(x) = ax^{a-1}$ series (6) reduces to solutions given by Grady and Ziff [11]. The solution is asymptotically lognormal [8] for any $B(x)$. Thus, for the interpretation of (2), if the impact rate and the probability $\nu(n, x)$ do not depend on x or n the fragment size distribution is asymptotically lognormal.

The solution (6) is a special case of

$$f(x,t) = \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-qt}(qt)^n}{n!} \boldsymbol{B}_n(x) = \sum_{n=0}^{\infty} k(n,t) \boldsymbol{B}_n(x). \tag{7}$$

Here $\boldsymbol{B}_n(x)$ is the size distribution for fragments that have undergone n fragmentation events, and $k(n, t)$ is the distribution of fragmentation number (the number of fragmentation events undergone). It is seen here that $k(n, t)$ is a Poisson distribution with parameter qt. Together with $N(t)$ from (5) it is a solution of the equation $\partial_t Nk(n, t) = qNk(n-1, t) - Nk(n, t)$. Thus, instead of seeking solutions in the x-description for different $B(y \to x)$ it is enough to solve this equation to obtain the solutions for all possible $\boldsymbol{B}_n(x)$. For $\alpha(x) = 1$ the $\boldsymbol{B}_n(x)$ are in general given by

$$\boldsymbol{B}_0(x) = f_0(x) \qquad \boldsymbol{B}_n(x) = \int_x^1 \mathrm{d}y\, \boldsymbol{B}_{n-1}(y) B_{n-1}(y \to x). \tag{8}$$

Here $B_n(y \to x)$ is the transition probability for fragments with fragmentation number n. For any $\boldsymbol{B}_n$, (7) is a distribution for a possible fragmentation process for $\alpha(x) = 1$. However, in the general case (8) it cannot be described by equation (1) as this requires that $B_n(y \to x)$ does not depend on n. If this condition holds and moreover $B(y \to x)\,\mathrm{d}x \equiv B(x/y)(\mathrm{d}x/y)$, (6) is obtained.

It is not strictly known whether $f(x)$ is asymptotically lognormal in the general case (8). The Poisson distribution approaches normal distribution as t grows large; this turns the fragmentation number n into a continuous variable and the probability for n being small becomes vanishingly small. For $B(z \to x)\,\mathrm{d}x \equiv B(x/z)(\mathrm{d}x/z)$, the $\boldsymbol{B}_n(x)$ is a distribution of the product of n identically distributed random variables. Owing to the central limit theorem, it can be approximated by a lognormal distribution where n appears as a parameter. Integration of these lognormals over the normal produces a distribution that can be shown to be asymptotically lognormal. As the conditions of the central limit theorem can be much relieved so that the lognormality of $\boldsymbol{B}_n(x)$ still holds, it is probable that the lognormal asymptotics is a very general property of constant rate processes.

2. Fragmentation equation including fragmentation number

In general all of the components α, B and κ can depend on the fragmentation number n, fragment size x, and time t. The number of fragments with fragmentation number n and size x is given by

$$\mu(n, x, t) = N(t) k(n, t) \boldsymbol{B}_n(x, t) \tag{9}$$

where $N(t)$ is the total number of fragments, $k(n, t)$ is the fragmentation number distribution and $\boldsymbol{B}_n(x, t)$ is the size distribution for fragments with fragmentation number n. In general $\boldsymbol{B}_n$ cannot be obtained iteratively through (8) as in the case $\alpha = 1$. Divided by $N(t)$, (9) is the distribution of the random vector (N_f, X), where N_f is the fragmentation number and X is the size: $\mu(n, x)\,\mathrm{d}x/N = \mathrm{Prob}(N_f = n \text{ and } x \leqslant X < x + \mathrm{d}x)$. The equation for μ is formulated as

$$\frac{\partial \mu(n,x)}{\partial t} = \int_x^1 \mathrm{d}y\, \alpha(n-1, y)\mu(n-1, y) B_{n-1}(y \to x)\kappa(n-1, y) - \alpha(n, x)\mu(n, x). \tag{10}$$

Here $\alpha(n, y, t)$ is the fragmentation rate and $B_n(y \to x; t)$ is the transition probability for (n, y)-fragments. The expectation of $B_n(y \to x; t)$ is $y/\kappa(n, y, t)$ and for q-ary fragmentation $\kappa = q$. If a solution (9) of (10) is known the following are obtained:

$$N(t) = \int_0^1 \mathrm{d}x \sum_{n=-0}^{\infty} \mu(n, x, t) \tag{11}$$

$$f(x, t) = N(t)^{-1} \sum_{n=0}^{\infty} \mu(n, x, t) = \sum_{n=0}^{\infty} k(n, t) \boldsymbol{B}_n(x, t) \tag{12}$$

$$k(n, t) = N(t)^{-1} \int_0^1 \mathrm{d}x\, \mu(n, x, t) = \int_0^1 \mathrm{d}x\, k(n, t) \boldsymbol{B}_n(x, t) \tag{13}$$

$$\boldsymbol{B}_n(x, t) = \frac{\mu(n, x, t)}{N(t) k(n, t)} \tag{14}$$

$$k_x(n, t) = \frac{\mu(n, x, t)}{N(t) f(x, t)}. \tag{15}$$

Here (11) is the total number of fragments, (12) is the fragment size distribution, (13) is the fragmentation number distribution, (14) is the fragment size distribution conditioned on fragmentation number n, and (15) is the fragmentation number distribution conditioned on fragment size x. If the components α and B of (10) do not depend on n, summation over all n gives the ordinary fragmentation equation (1). If α and κ do not depend on x integration gives the fragmentation number equation

$$\frac{\partial N(t) k(n, t)}{\partial t} = \kappa(n-1)\alpha(n-1) N(t) k(n-1, t) - \alpha(n) N(t) k(n, t) \tag{16}$$

in which case, provided that $B_n(y \to x)$ are time-independent, $\boldsymbol{B}_n$ are given by (8).

3. Solutions

3.1. Series solutions

Equation (10) is an iterative first-order differential equation for $\mu(n, x, t)$ and in principle completely integrable even when all components of the equation are time dependent. Starting from N_0 $(n = 0, x = 1)$-fragments only the second term on the right counts and

$$\mu(0, x, t) = N_0 \delta(1 - x) \exp\left\{ -\int_0^1 \mathrm{d}t\, \alpha(0, 1, t) \right\} \tag{17}$$

is the number $(n = 0, x = 1)$-fragments persisting at time t. This solution is inserted into equation (10) for $\mu(1, x, t)$ which can then be integrated, and the procedure is iterated for all $\mu(n, x, t)$. In practice the cases which can be carried through are usually more directly obtainable by other methods. Solutions for all n with $\alpha(n, x)$ depending both on n and x are generally unknown. However, for cases where n is limited manageable solutions are feasible. In many forced fragmentation processes a small number of fragmentations is sufficient to attain the desired degree of diminution.

3.2. Fragmentation number distribution derived from a fragment size distribution

If the solution $f(x,t)$ of (1) for any q-ary fragmentation process ($\kappa = q$) is known, the distributions $k(n,t)$ and $\boldsymbol{B}_n(x,t)$ pertaining to the process can be derived. Defining

$$S(p,x,t) = \sum_{n=0}^{\infty} p^n k(n,t)\boldsymbol{B}_n(x,t) \tag{18}$$

the following equation is obtained by multiplying (10) by p^n and summing over n:

$$\frac{\partial NS(p,x,t)}{\partial t} = Npq\int_x^1 \mathrm{d}y\,\alpha(y)B(y\to x)S(p,y,t) - \alpha(x)S(p,x,t). \tag{19}$$

In comparison with (1) it is seen that $S(p,x,t)$ can be obtained from $f(x,t)$ by substituting pq to every instance of q. The fragmentation number distribution is then derivable as

$$k(n,t)\boldsymbol{B}_n(x,t) = (\partial^n S(p,x,t)/\partial p^n)_{p=0} \qquad k(n,t) = \int_0^1 \mathrm{d}x\,k(n,t)\boldsymbol{B}_n(x,t) \tag{20}$$

from which $\boldsymbol{B}_n(x,t)$ are also obtained.

For a special case $\alpha(x) = x^\rho$, $B(z\to x)\,\mathrm{d}x \equiv B(x/z)(\mathrm{d}x/z)$, transform (3) with the kernel $x^{\rho s}$ reduces (1) to a differential–difference equation $\partial_1 NF(s,t) = NF(s+1,t)(qD(s)-1)$.

The solution can formally be written as a series. If $\boldsymbol{D}_n(s,t)$ is the transform of $\boldsymbol{B}_n(x,t)$, and

$$S(p,s,t) = \int_0^1 \mathrm{d}x\,x^{\rho s}S(p,x,t) = \sum_0^{\infty} p^n k(n,t)\boldsymbol{D}_n(s,t) \tag{21}$$

$S(p,s,t)$ is obtained from $F(s,t)$ by substituting pq to every instance of q and

$$k(n,t)\boldsymbol{D}_n(s,t) = \frac{\partial^n S(p,s,t)}{\partial p^n}\Big|_{p=0} \qquad k(n,t) = k(n,t)\boldsymbol{D}_n(0,t) \tag{22}$$

which then give $\boldsymbol{D}_n(s,t)$. As an example, restricting us further to $B(x) = 1$, the series solution of the differential–difference equation is recognized as

$$NF(s,t) = M(s-(q-1)/\rho, s+1/\rho, -t)$$
$$N(t) = N(t)F(0,t) = M(-1/\rho, 1/\rho, -t)$$

where M is a confluent hypergeometric function and $q = 2$. The inverse, or solution $f(x,t)$, can be found in [11]. Replacing q with $2p$, $NS(p,s,t) = M(s-(2p-1)/\rho, s+1/\rho, t)$ and from (22)

$$k(n,t) = N(t)^{-1}\partial_p^n M((2p-1)\rho, 1/\rho, t)|_{p=0}. \tag{23}$$

The integral representation of M can be used to derive a representation for $k(n)$ at least when $\rho = 1$.

3.3. Solving the fragmentation number equation

For the initial conditions $k(n,0) = k_0(n)$ and $N(0) = N_0$ the $k(n,t)$ and $N(t)$ are obtained from (16) by solving for $N(t)k(n,t)$ and normalizing to 1. However, it is seen that for any $k_0(n)$ the solution of (16) can be written as follows

$$N(t)k(n,t) = \sum_{m=0}^{n} N_0 k_0(m) N_m(t) k_m(n,t) \tag{24}$$

where $N_m(t)k_m(n,t)$ $(n = m, m+1, \ldots)$ is the solution of (16) for the initial condition of a single m-fragment, $N_m(0)k_m(n,0) = \delta_{mn}$. Defining

$$\begin{aligned} N_m(t)k_m(n,t) &= \mathrm{e}^{-\alpha(n)t}\kappa(m)\kappa(m+1)\ldots\kappa(n-1)\frac{\alpha(m)\alpha(m+1)\ldots\alpha(n-1)}{\beta(m)\beta(m+1)\ldots\beta(n)} \\ &\quad\times N_m(t)k_m^*(n,t) \qquad n > m \\ N_m(t)k_m(m,t) &= \mathrm{e}^{-\alpha(m)t} \qquad N_m(t)k_m^*(m,t) = 1 \\ N_m(t)k_m(n,t) &= 0 \qquad n < m \end{aligned} \tag{25}$$

where $\beta(n)$ is arbitrary, (16) yields

$$\frac{\partial N_m(t)k_m^*(n,t)}{\partial t} = \beta(n)\mathrm{e}^{\alpha(n)t-\alpha(n-1)t}N_m(t)k_m^*(n-1,t). \tag{26}$$

It is seen that the solution of (26) gives solution (16) for all possible functions $\kappa(n)$. The solution has the form

$$N_m(t)k_m^*(n,t) = \mathrm{e}^{\alpha(n)t}(c_n^n\mathrm{e}^{-\alpha(n-1)t} + c_{n-1}^n\mathrm{e}^{-\alpha(n-2)t} + \cdots + c_{m+1}^n\mathrm{e}^{-\alpha(m)t}) + c_m^n. \tag{27}$$

This is substituted into (26) and the coefficients are solved. Written in terms of a $(n-m+1)\times(n-m)$ matrix and the shorthand $\beta(n)(\alpha(n)-\alpha(k))^{-1} = \Delta_k^n$, this gives

$$\begin{bmatrix} c_m \\ c_{m+1}^n \\ c_{m+2}^n \\ \vdots \\ c_{n-2}^n \\ c_{n-1}^n \\ c_n^n \end{bmatrix} = \begin{bmatrix} -\Delta_{n-1}^n & -\Delta_m^n & -\Delta_{m+1}^n & \ldots & -\Delta_{n-3}^n & -\Delta_{n-2}^n \\ 0 & \Delta_m^n & 0 & \ldots & 0 & 0 \\ 0 & 0 & \Delta_{m+1}^n & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \Delta_{n-3}^n & 0 \\ 0 & 0 & 0 & \ldots & 0 & \Delta_{n-2}^n \\ \Delta_{n-1}^n & 0 & 0 & \ldots & 0 & 0 \end{bmatrix} \times \begin{bmatrix} c_m^{n-1} \\ c_{m+1}^{n-1} \\ c_{m+2}^{n-1} \\ \vdots \\ c_{n-2}^{n-1} \\ c_{n-1}^{n-1} \end{bmatrix} \tag{28}$$

which is then iterated from $c_m^m = 1$. The factor $\beta(n)$ can be suitably chosen to simplify the expressions.

3.4. Fragment size distributions derivable from a fragmentation number distribution

The fragment size distribution $f(x,t)$ can be approached from a known solution $k(n,t)$ of (16) in two ways. As for any time-independent $B_n(x \to y)$ the $\boldsymbol{B}_n(x)$ are given by (8) the corresponding $f(x,t)$ can be constructed as series (12). If $B_n(x \to y)$ does not depend on n this distribution is then a solution of (1) with the rate

$$\alpha(x,t) = \sum_{n=0}^{\infty}\alpha(n)k_x(n,t) = \frac{1}{f(x,t)}\sum_{n=0}^{\infty}\alpha(n)k(n,t)\boldsymbol{B}_n(x). \tag{29}$$

Thus it is, in principle, possible to construct endlessly new solutions of the fragmentation equation (1) using this procedure; in practice only a few series (12) have a simple closed form.

Any solution $k(n,t)$ also provides in a more direct manner a solution of the ordinary fragmentation equation (1) for cases where the progeny fragments are of the same size. For q-ary fragmentation $(\kappa = q)$, $B(x) = \delta(x - 1/q)$, and starting from $(n = 0, x = 1)$-fragments, fragmentation number n corresponds to size $(1/q)^n$ and $\mathrm{Prob}(X = (1/q)^n) = \mathrm{Prob}(N_f = n)$. As $\alpha(n) = \alpha(\ln(1/q)^n/\ln(1/q))$, for each solution $k(n,t)$ there corresponds a solution $f(x,t)$ of the fragmentation equation (1) with $B(x) = \delta(x - 1/q)$ and $\alpha(x)$ obtained by substituting $\ln(x)/\ln(1/q)$ for n in $\alpha(n)$. However, $\kappa(x)$ may also depend on x (or, equivalently, on n) so that the progeny fragments are of equal size but the number of progeny may change. Then $n(x)$ should be solved from $x(n) = (\kappa(0)\kappa(1)\ldots\kappa(n-1))^{-1}$. If this can be done, $\kappa(x)$ is obtained by substituting $n(x)$ to $\kappa(n)$, and $B(y \to x) = \delta(x - y/\kappa(n(y)))$ is a progeny distribution dependent on parent size y. The equivalence also works of course the other way round. Thus it is seen that the solutions of (1) for $B(x) = \delta(x - 1/q)$ and $\alpha(x) = x^\rho$, ρ any real number, can be obtained by solving the fragmentation number equation (16) for rate $\alpha(n) = \exp(an)$, a any real number. It is, however, seen from (27) and (28) that these solutions are very complicated.

3.5. Linear dependence on fragmentation number

For $\alpha(n) = n + a, a > 0$, and q-ary fragmentation ($\kappa = q$) the fragmentation number equation

$$\frac{\partial Nk(n,t)}{\partial t} = qN(t)(n + a - 1)k(n-1,t) - N(t)(n+a)k(n,t) \tag{30}$$

reduces with the substitutions $k(n,t) = \mathrm{e}^{-(n+a)t}q^n a(a+1)\ldots(n+a-1)k^*(n,t)$, $\mathrm{e}^t = \tau$ to

$$\frac{\partial N(\tau)k^*(n,\tau)}{\partial \tau} = N(\tau)k^*(n-1,\tau). \tag{31}$$

For the initial condition $Nk^*(0,1) = N_0$, $Nk^*(n,1) = 0$ for $n > 0$, the solution is $N(\tau)k^*(n,\tau) = N_0(\tau-1)^n/n!$. Thus for the initial condition of the number N_0 of fragments with fragmentation number zero,

$$N(t)k(n) = N_0\mathrm{e}^{-at}a(a+1)\ldots(n+a-1)\frac{q^n}{n!}(1-\mathrm{e}^{-t})^n \tag{32}$$

from which, defining $p = 1 - q + q\mathrm{e}^{-t}$,

$$k(n,t) = p^a a(a+1)\ldots(n+a-1)\frac{1}{n!}(1-p)^n \qquad N(t) = N_0 p^{-a}\mathrm{e}^{-at}. \tag{33}$$

Thus, $k(n,t)$ is a negative binomial distribution with parameters a and p. The geometric distribution is obtained for $a = 1$. As $\kappa > 1$, thus also in cases (2) where n is taken to be the impact number, the process reaches in a finite time $t_\infty = \ln(\kappa/\kappa - 1)$ a stage where $p = 0$ and an infinite number of infinitesimal fragments is created. This corresponds to 'shattering' found for ordinary fragmentation equation solutions when $\alpha(x) \sim x^\rho$, $\rho < 0$ [2, 3, 11, 12]. It is easily seen that the solution with initial condition δ_{nm} is $k_m(n,t) = k(n-m,t; a+m)$, $n = m, m+1, \ldots,$ where $k(n,t; a+m)$ is solution (33) with parameters $a+m$ and p. From these any solution (24) can be constructed.

The fragment size distributions (12) derivable from (33) are studied here for the initial situation of $(n = 0, x = 1)$-fragments, $k(n,0) = \delta_{n0}$, $f(x,0) = \delta(x-1)$, and for $B(z \to x)\,\mathrm{d}x \equiv B(x/z)(\mathrm{d}x/z)$ where $B(x)$ has the transform $D(s)$, (3). The transform of $\boldsymbol{B}_n(x)$, (8), is $D(s)^n$ and the transform of $f(x,t)$ is obtained by inserting $k(n,t)$ from (33) into (12) as

$$F(s,t) = \frac{p^a}{(1-(1-p)D(s))^a}. \tag{34}$$

This gives the moment of $f(x,t)$ of any order n for $s = n$. For $B(x) = 1$ (uniform progeny distribution), $\boldsymbol{B}_n(x) = (1/(n-1)!)\ln^{n-1}(1/x)$ and the series (12) for the fragment size distribution can be expressed in terms of a confluent hypergeometric function M as

$$f(x,t) = p^a\delta(x-1) + a(1-p)p^a M(a+1, 2, (1-p)\ln(1/x)). \tag{35}$$

For $a = 1$ this reduces to

$$f(x,t) = p\delta(x-1) + (1-p)px^{-(1-p)}. \tag{36}$$

The fragment size distribution (36) is a decreasing power function with a time-dependent exponent. It approaches $\sim x^{-1}$ as $t \to t_\infty$. The appearance of power laws is generally understood as one universal feature of the fragmentation processes. It can be derived using fractal fragmentation models where a certain portion of the smallest fragments are disintegrated further while the rest remain intact hereafter [15]. It also follows from more complicated discrete and combinatorial approaches [1, 9, 10]. In certain continuous fragmentation processes it is found asymptotically and for a small fragment regime [2, 3, 12]. Result (36) shows that the power law can also be a proper solution for the continuous fragmentation equation where the exponent can be a function of time. The rate for which (36) is the solution of (1) can be calculated by (29). In the asymptotical regime, where the delta function details can be neglected, this rate is $\alpha(x) = 2 - (1-p)\ln x$. Thus, a solution of (1) with logarithmically behaving rate became constructed.

From solution (33) a solution of the fragmentation equation (1) for q-ary fragmentation into equal pieces $B(x) = \delta(x - 1/q)$ is obtained by the equivalence $\mathrm{Prob}(X = (1/q)^n) = \mathrm{Prob}(N_f = n)$. Thus $\alpha(x) = \ln(x)/\ln(1/q) + a$. For the geometric distributions case $a = 1$, $k(n,t) = p(1-p)^n$, the number of fragments with size x is then proportional to $x^{\ln(1-p)/\ln(1/\kappa)}$. The exponent decreases from infinity at $t = 0$ to zero in the finite time t_∞. Thus the power law is found again but with a positive exponent. As the production of small fragments is in its minimum when $B(x)$ is a delta function, shattering is expected whenever $\alpha(x) \sim \ln(x)$.

3.6. Inversely proportional case

For $\alpha(n) = 1/(n+a)$, $a \geqslant 0$, it is chosen $\beta(n) = \alpha(n)$ so that

$$\Delta_n^k = \beta(n)\left(\frac{1}{n+a} - \frac{1}{k+a}\right)^{-1} = -\frac{k+a}{n-k}. \tag{37}$$

Iterating (28) for the initial condition $Nk^*(0,1) = N_0$, $Nk^*(n,1) = 0$ for $n > 0$, the solution emerges as

$$N(t)k^*(n,t) = N_0 \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^k \mathrm{e}^{-\frac{k}{(n+a)(n-k+a)}t}(n-k+a)^{n-1} \tag{38}$$

and from (25) for q-ary fragmentation ($\kappa = q$)

$$\begin{aligned} N(t)k(n,t) &= N_0 q^n \mathrm{e}^{-\frac{1}{n+a}t}(n+a)\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^k \mathrm{e}^{-\frac{k}{(n+a)(n-k+a)}t}(n-k+a)^{n-1} \\ &= N_0 q^n (n+a)\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\mathrm{e}^{-\frac{1}{k+a}t}(k+a)^{n-1}. \end{aligned} \tag{39}$$

For any time instant t it is found that negative values $N(t)k(n,t)$ start to appear after a certain value of n. As this cannot be, the support of $N(t)k(n,t)$ is finite for all t. Summation over the support gives $N(t)$, and $k(n,t)$ obtained thereby sums to unity over the support. The finite support is in a way a phenomenon opposite to shattering.

Solution (39) immediately gives a more general one. For the rate $\alpha(n) = (an+b)/(cn+d)$ the choice

$$\beta(n) = \frac{b/c - da/c^2}{n + d/c} \qquad \tau = (b/c - da/c^2)t \tag{40}$$

produces an equation for $k^*(n,\tau)$ with solutions obtained from (38) by changing a to d/c and t to τ. This gives the solution

$$N(t)k(n,t) = N_0 \frac{q^n \mathrm{e}^{-at/c}}{(bc-ad)^n} b(a+b)\ldots(a(n-1)+b) \times \frac{cn+d}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \mathrm{e}^{-\frac{(b-da/c)}{ck+d}t} (ck+d)^{n-1}. \tag{41}$$

The rate $\alpha(n)$ is increasing, decreasing or constant depending on whether $ab - bc$ is positive, negative or zero; in the third case the Poisson distribution appearing in (7) is obtained as a limit. Solutions (33) and (39) are obtained by a proper choice of constants a, b, c and d. Through $\mathrm{Prob}(X = (1/q)^n) = \mathrm{Prob}(N_f = n)$ the solution corresponds to a $B(x) = \delta(x - 1/q)$ solution of (1) with the rate $\alpha(x) = \frac{a\ln(x)+b\ln(1/q)}{c\ln(x)+d\ln(1/q)}$. This rate can be chosen to be finite for all $0 \leqslant x \leqslant 1$, having then the value b/d for fragment size 1 and value a/c for small fragment limit.

3.7. Progeny production dependent on n

From any solution $k(n,t)$ for q-ary fragmentation, especially from (41), an indefinite number of other solutions corresponding to different choices of $\kappa(n)$ can be generated by replacing q^n in the solution by $\kappa(0)\kappa(1)\ldots\kappa(n-1)$. Allowing the possibility of interpreting n as an impact number it is chosen

$$\kappa(n) = 1 - \nu(n) + \nu(n)\kappa^*(n) \tag{42}$$

where $\nu(n)$ is the probability of fragmenting in the nth impact and $\kappa^*(n)$ the number of progeny produced in this case.

As an example the case $\alpha(n) = 1$, υ constant and $\kappa^*(n) = n + 2$ is studied so that $\kappa(n) = \nu(n + 2 + (1-\nu)/\nu)$. Thus the number of progeny increases linearly with the impact number. For q-ary fragmentation the solution for N_0 $(n = 0)$-fragments at $t = 0$ is $Nk(n) = N_0\mathrm{e}^{-t}(qt)^n/n!$ and by the replacement of q^n the following solution is obtained

$$N(t)k(n,t) = N_0\mathrm{e}^{-t}\frac{(\nu t)^n}{n!}\left(2+\frac{1-\nu}{\nu}\right)\left(3+\frac{1-\nu}{\nu}\right)\ldots\left(n-1+2+\frac{1-\nu}{\nu}\right). \tag{43}$$

Comparison with (32) reveals that $k(n,t)$ is a negative binomial distribution with the parameters $p = 1 - \nu t$ and $a = 2 - (1-\nu)/\nu$. Thus the process reaches the shattering phase at time $t = 1/\nu$. In this case shattering does not follow from the increasing rate but from the increasing number of progeny. Solutions $f(x,t)$ can again be constructed, and for $\nu = 1$ and $B(x) = 1$ the power law solution (36) is found. The limit distribution is thus not lognormal although the rate is constant.

4. Cluster size distributions in cellular structures

A large number of boxes is assumed so that the rate for objects to be put in a box follows a dependency $\alpha_c(n)$ on the number n of objects already in the box; the subscript c is added to avoid confusion with the fragmentation number. The equation governing the distribution of numbers of objects in the boxes is

$$\frac{\partial k_c(n,t)}{\partial t} = \alpha_c(n-1)k_c(n-1,t) - \alpha_c(n)k_c(n,t) \tag{44}$$

or the fragmentation number equation with constant N and $\kappa = 1$.

The box process can also be interpreted to describe fragmentation. At $t = 0$ an initial situation is assumed where, say, a unit square has been fragmented in an arbitrary fashion so that the pieces keep their positions. The initial situation defines the boxes of the process; the fragment boundaries can be imagined to become permanently marked. The fragmentation continues in a binary fashion, generating a certain cellular structure. The new fragmentation lines are objects in the boxes. Their number is n^* so that $n = n^* + 1$ is the number of fragments in the box (including the box itself). If the rate $\alpha_c(n^*)$ for fragmentation events to occur in a cell depends on the number $n^* + 1$ of fragments in a cell, the process is governed by (44) with the initial condition $k_c(n^*, 0) = \delta_{n^*0}$. The boxes can be interpreted as clusters and n as cluster size; this gives a model where the cluster size increase rate is $\alpha_c(n^*)$. For increasing $\alpha_c(n^*)$ the clusters attract fragmentation events, for decreasing $\alpha_c(n^*)$ repel them.

This has an important application for the process where the fragmentation rate $\alpha(x, n)$, in its usual interpretation, does not depend on x or n. This produces size distributions (7) which are asymptotically lognormal at least if $B(z \to x)\,\mathrm{d}x \equiv B(x/z)(\mathrm{d}x/z)$. As then each fragment in a cluster is fragmented with equal rate with unit magnitude, the overall rate $\alpha_c(n^*)$ of fragmentations to occur in the cluster is $n^* + 1 = n$. The distribution $k_c(n^*, t)$ of the number of fragmentation lines in a cluster is governed by equation (44) and the solution is given by (33). As the average number of fragments per cluster is $N_C(t) = \mathrm{e}^t$, and $n^* = n - 1$, the probability of having n fragments in a cluster is geometrically distributed and obtained from (33) with parameters $a = 1$, $\kappa = 1$ and $p = (1/N_C)$,

$$k_c(n, t) = \frac{1}{N_C(t)} \left(1 - \frac{1}{N_C(t)}\right)^{n-1} \qquad n = 1, 2, \ldots. \tag{45}$$

Thus for any partition defining the cluster boundaries at $t = 0$, the constant rate fragmentation process leads to the cluster size distribution (45) where N_C is the average cluster size.

From the properties of geometric distribution it follows that if we choose a minimum size n_0 for a cluster, they are distributed as $k_c(n|n \geqslant n_0) = k_c(n - n_0 + 1|n \geqslant 1)$. The size of sums of m clusters has negative binomial distribution (33) with $a = m, \kappa = 1$ and $p = 1/N_C$. Further, assume that at time $t_1 = \ln(N_{C1})$ the existing clusters are defined as first-order clusters and all existing fragments are taken to define the boundaries of second-order clusters. The first-order cluster is thus composed of the second-order clusters. Then at time $t_2 = \ln(N_{C2})$ the distribution for first-order clusters is geometric with parameter $1/N_{C1}$ and for second-order clusters geometric with parameter $1/N_{C2}$. The combined clusters, understood as containing all fragments inside the first-order cluster boundaries, is geometric with parameter $1/N_{C1}N_{C2}$. In the limit of large t this hierarchy can be iterated indefinitely.

In [8] it was shown that the process with size-independent fragmentation rate produces multivariate lognormal distributions and the correlations between neighbouring fragments were given. This theory was formulated for the fragmentation of a unit line and the results were not readily adaptable to other geometries, owing to the ambiguity of the neighbouring concept. Result (45) can be considered as an adjoint way of attacking the problem, being easily applicable to any geometry. From the constant fragmentation rate there is a relation between $\alpha(x)$ and $\alpha_c(n)$. Another such case is $\alpha(x) = x$, or the spatial Poisson process, but for other rates $\alpha(x)$ the cluster size equation (44) is an approach independent of the fragmentation equation (1). In the case where the cells in the initial partitioning at $t = 0$ have the same size (44) can be understood as parametrized by the average size of the fragments in the cluster.

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